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Closures of Conjugacy Classes in G_2

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1. Introduction

1.1. In a recent paper [LS] Levasseur and Smith have shown that the 8-dimensional nilpotent conjugacy class in the simple Lie algebra $\mathfrak{g}$ of type G_2 has a non-normal closure. In the following we give a short proof of this result and show that all other classes have a normal closure. This completes the geometric picture in the spirit of the work [KP1, KP2, KP3], which deals with the case of the classical groups.

1.2. THEOREM.[1] *Let $\mathfrak{g}$ be the simple Lie algebra of type G_2 and let C_i be the nilpotent conjugacy class in $\mathfrak{g}$ of dimension $i = 6$, 8, 10, and 12.*

(a) *Every conjugacy class of $\mathfrak{g}$ except C_8 has a normal closure with rational singularities.*

(b) *[Levasseur and Smith] $\overline{C_8}$ is not normal in $\overline{C_6} = \overline{C_8} \setminus C_8$. The normalization $\eta_8 : \widetilde{C_8} \to \overline{C_8}$ is bijective and $\widetilde{C_8}$ has an isolated rational singularity in $\eta_8^{-1}(0)$.*

(c) *$\overline{C_{12}}$ has a singularity of type D_4 in C_{10}, and $\overline{C_{10}}$ a singularity of type A_1 in C_8.*

The proof of these results is based on the same construction as in [LS]: We embed $\mathfrak{g}$ into $\mathbf{so}_7$ by the 7-dimensional standard representation, and study the $\mathfrak{g}$-equivariant projection $p: \mathbf{so}_7 \to \mathfrak{g}$. It turns out that the restriction of p to the 8- and 10-dimensional nilpotent conjugacy classes D_8 and D_{10} in $\mathbf{so}_7$ induces finite surjective morphisms

$$p_8 : \overline{D_8} \to \overline{C_8} \qquad \text{and} \qquad p_{10} : \overline{D_{10}} \to \overline{C_{10}}.$$

The result then follows from a careful analysis of these two maps.

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[1] In a preliminary version of this paper under the title "Non-normality of Closures of Conjugacy Classes in G_2" the statement of this theorem is not correct.

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1.3. *Remark.* The simple group G_2 also has an exceptional behavior with respect to the *sheets* in its Lie algebra $\mathbf{g}$ (cf. [Kr1], [BK], or [B] for definitions). One of the two subregular sheets through C_{10} is smooth, the other is not normal and C_{10} undergoes a threefold covering in the normalization of that sheet [Sl, p. 151; BK, 7.3 Beispiel b].

Conventions. The base field k is algebraically closed of characteristic zero. If X is an algebraic variety we denote by $k[X]$ the algebra of global regular functions on X.

2. Nilpotent Conjugacy Classes in G_2

2.1. Let G be a simple group of type G_2. Fix a maximal torus T and a Borel subgroup $B \supset T$ and denote by α_1, α_2 the corresponding base of the root system Φ with respect to T, where α_1 is a short and α_2 a long root. The nilpotent cone of the Lie algebra $\mathbf{g} := \operatorname{Lie} G$ consists of five conjugacy classes C_{12}, C_{10}, C_8, C_6, and $C_0 = \{0\}$ with dimensions $\dim C_i = i$. C_6 is the conjugacy class of a long root vector $x_2 \in \mathbf{g}_{\alpha_2}$, C_8 the class of a short root vector $x_1 \in \mathbf{g}_{\alpha_1}$, C_{10} the class of $x_2 + x_2'$, where $x_2' \in \mathbf{g}_{3\alpha_1+\alpha_2}$ is another long root vector, and C_{12} the class of $x_1 + x_2$. C_{12} is the *regular class*, C_{10} the *subregular class*, and we have $\overline{C_i} \supset C_j$ for $i \geq j$. (Cf. [SK])

2.2. The long root vectors generate the subalgebra

$$\operatorname{Lie} T \oplus \bigoplus_{\beta \text{ long}} \mathbf{g}_\beta$$

of $\mathbf{g}$ which we will identify with $\mathbf{sl}_3$. It is easy to see that $\mathbf{g}$ decomposes as an $\mathbf{sl}_3$-module in the form

$$\mathbf{g} = \mathbf{sl}_3 \oplus k^3 \oplus (k^3)^*.$$

We can therefore consider SL_3 as a subgroup of G.

2.3. *Remark.* Let $H := G_x$ be the centralizer of $x := x_2 + x_2' \in C_{10}$. It is known that H^0 is unipotent and that $H/H^0 \simeq \mathcal{S}_3$, the symmetric group in three letters (see [Ca, table on p. 401]). We will only need that H contains the center $Z \simeq \mathbf{Z}/3\mathbf{Z}$ of SL_3 and that

$$\operatorname{Lie} H \supset \mathbf{g}_{\alpha_1+\alpha_2} \oplus \mathbf{g}_{2\alpha_1+\alpha_2} \oplus \mathbf{g}_{3\alpha_1+2\alpha_2};$$

this is obvious from what we have said above.

2.4. *Remark.* The centralizer of $x_1 \in C_8$ is connected (see [Ca, table on p. 401]).

3. The Standard Representation of G_2

3.1. Let $\rho: G \to GL(V)$ be the 7-dimensional irreducible representation with highest weight $\omega_1 = 2\alpha_1 + \alpha_2$. This representation is orthogonal, and the invariant quadratic form $q: V \to k$ generates the algebra $k[V]^G$ of G-invariant polynomials on V. The zero cone $V^\circ := q^{-1}(0)$ consists of two orbits, $\{0\}$ and Gv_0, where $v_0 \in V_{\omega_1}$ is a highest weight vector. (As usual we denote by V_β the weight space of V of weight β.)

The representation ρ defines an embedding $G \hookrightarrow SO_7$. The adjoint representation of G on $\mathbf{so}_7$ is isomorphic to $\bigwedge^2 V$ and decomposes in the form

$$\mathbf{so}_7 \simeq \bigwedge^2 V \simeq \mathbf{g} \oplus V.$$

(This is clear from dimensional reason.)

The weights of V are the zero weight and the short roots of $\mathbf{g}$, all with multiplicity one. Hence we have

$$V \simeq k \oplus k^3 \oplus (k^3)^*$$

as an SL_3-module.

3.2. Lemma. *Let $v_0 \in V \subset \mathbf{so}_7$ be a highest weight vector of V. As an element of $\mathbf{so}_7$ the endomorphism v_0 is nilpotent with partition* $(3, 2, 2)$.

(The *partition* of a nilpotent endomorphism is given by the sizes of the blocks in a Jordan normal form.)

Proof. The highest weight of V is $\omega_1 := 2\alpha_1 + \alpha_2$, and the corresponding weight space in $\bigwedge^2 V$ is 2-dimensional:

$$\left(\bigwedge^2 V\right)_{\omega_1} = V_0 \wedge V_{\omega_1} \oplus V_{\alpha_1} \wedge V_{\alpha_1+\alpha_2}.$$

Since a highest weight vector $v_0 \in V \subset \bigwedge^2 V$ is annihilated by $\mathbf{g}_{\alpha_1}$ and $\mathbf{g}_{\alpha_2}$ we see that v_0 has non-zero components in both summands:

$$v_0 = w_0 \wedge w_2 + w_1 \wedge w_3,$$

$$w_0 \in V_0,\ w_1 \in V_{\alpha_1},\ w_2 \in V_{2\alpha_1+\alpha_2},\ w_3 \in V_{\alpha_1+\alpha_2} \text{ and all } w_i \neq 0.$$

Furthermore, the G-isomorphism $\sigma: V \xrightarrow{\sim} V^*$ satisfies $\sigma(V_\beta) = (V_{-\beta})^*$ for all weights β. It follows that the composition

$$\bigwedge^2 V \hookrightarrow V \otimes V \xrightarrow{\sim} V \otimes V^* \xrightarrow{\sim} \operatorname{End} V$$

maps the element

$$w_0 \wedge w_2 + w_1 \wedge w_3 = \tfrac{1}{2}(w_0 \otimes w_2 - w_2 \otimes w_0 + w_1 \otimes w_3 - w_3 \otimes w_1)$$

to an element of the form

$$w = w_0 \otimes \bar{w}_{-2} + w_2 \otimes \bar{w}_0 + w_1 \otimes \bar{w}_{-3} + w_3 \otimes \bar{w}_{-1}$$

with non-zero elements $\bar{w}_0 \in (V_0)^*$, $\bar{w}_{-1} \in (V_{-\alpha_1})^*$, $\bar{w}_{-2} \in (V_{-2\alpha_1-\alpha_2})^*$, $\bar{w}_{-3} \in (V_{-\alpha_1-\alpha_2})^*$. It is now easy to see that w is a nilpotent element of $\operatorname{End} V$ with partition (3, 2, 2). ∎

4. The Fundamental Construction

4.1. The nilpotent cone of $\mathbf{so}_7$ consists of the conjugacy classes D_{18}, D_{16}, D_{14}, D_{12}, D_{10}, D_8, and $D_0 = \{0\}$ of dimensions $\dim D_i = i$. They are completely determined by their conjugacy class in $M_7(k)$ with respect to $GL_7(k)$ and correspond to the partitions (7), (5, 1, 1), (3, 3, 1), (3, 2, 2), $(3, 1^4)$, $(2, 2, 1^3)$, and (1^7). (For this and the following see [KP3, Sect. 19 tables].) We have $\overline{D_i} \supset D_j$ for $i \geqslant j$, and all D_i except D_{12} have a normal closure $\overline{D_i}$ with rational singularities.

Consider the G-linear projection $p\colon \mathbf{so}_7 \to \mathbf{g}$ given by the decomposition $\mathbf{so}_7 = \mathbf{g} \oplus V$ (see Section 3).

4.2. Proposition. *The map p induces finite surjective morphisms*

$$p_8\colon \overline{D_8} \to \overline{C_8} \qquad \textit{and} \qquad p_{10}\colon \overline{D_{10}} \to \overline{C_{10}}.$$

The morphism p_8 is bijective, but it is not an isomorphism in the points of $p_8^{-1}(\overline{C_6})$ (i.e., the fibres over these points are not reduced).

Proof. By the lemma above we have $\overline{D_{10}} \cap V = \overline{D_8} \cap V = \{0\}$. Since the closures $\overline{D_i}$ are closed G-stable cones in $\mathbf{so}_7$ it follows that the images $X_8 := p(\overline{D_8})$ and $X_{10} := p(\overline{D_{10}})$ are closed and G-stable cones in $\mathbf{g}$ and that the maps $p_8\colon \overline{D_8} \to X_8$ and $p_{10}\colon \overline{D_{10}} \to X_{10}$ are finite morphisms. In fact, given a finitely generated graded algebra $R = \bigoplus_i R_i$ with $R_0 = k$ and a graded subalgebra $S = \bigoplus_i S_i$ such that $\sqrt{RS_+} = R_+$, where R_+ and S_+ are the homogeneous maximal ideals of R and S, then R is a finitely generated S-module (see, for example, [Kr2, II.4.3 Satz 8] and its proof). But C_6 and C_8 are the only conjugacy classes in $\mathbf{g}$ of dimension 6 and 8, all other classes have dimension $\geqslant 10$ (see Section 2). Hence $X_8 = \overline{C_8}$, and X_{10} contains a dense 10-dimensional conjugacy class and so $X_{10} = \overline{C_{10}}$. Since the centralizer of $x_1 \in C_8$ is connected (Remark 2.4) the map p_8 is birational.

Let $x_2 \in C_6$ be a long root vector as in Section 2. We claim that $p_8^{-1}(x_2) = \{(x_2, 0)\} \subset \mathbf{g} \oplus V$. In particular $p_8^{-1}(\overline{C_6}) \to \overline{C_6}$ is bijective. In fact, consider the B-stable line $kx_2 \subset \mathbf{g}$. Then $p_8^{-1}(kx_2)$ is finite union of lines, each one stable under B. Since there are only two B-stable lines in $\mathbf{so}_7$, namely $k(x_2, 0)$ and the highest weight space of V, the claim follows.

Next we show that the fibre $p_8^{-1}(x_2)$ is not reduced. In fact, the fibre $p_8^{-1}(x_2)$ is the schematic intersection of D_8 with $\{x_2\} \times V$. For the tangent spaces we find

$$T_{(x_2,0)} D_8 = [x_2, \mathbf{so}_7] = [x_2, \mathbf{g}] \oplus [x_2, V].$$

Since $[x_2, V] \neq (0)$ the intersection $D_8 \cap (x_2, V)$ is not transversal, hence the fibre $p_8^{-1}(x_2)$ is not a reduced point. ▮

4.3. *Remark.* Every nilpotent class $C_i \subset \mathbf{g}$ generates a nilpotent class in $\mathbf{so}_7$, via the embedding $\mathbf{g} \subsetneq \mathbf{so}_7$ (3.1). From the explicit description of the nilpotent classes C_i in 2.1 it is easy to determine the nilpotent endomorphism of V induced by $x \in C_i$; e.g., a short root $x_2 \in \mathbf{g}$ defines an endomorphism with partition $(3, 2, 2)$ and a long root $x_2 \in \mathbf{g}$ one with partition $(2, 2, 1^3)$. Using 4.1 we find the following inclusions:

$$C_6 \subset D_8, \qquad C_8 \subset D_{12}, \qquad C_{10} \subset D_{14}, \qquad C_{12} \subset D_{18}.$$

5. Multiplicities

5.1. For any G-variety X we denote by $\mathrm{mult}_M(X)$ the multiplicity of an irreducible representation M in the algebra $k[X]$ of global regular functions on X, i.e.,

$$\mathrm{mult}_M(X) = \dim_k \mathrm{Mor}_G(X, M^*),$$

where $\mathrm{Mor}_G(X, M^*)$ is the k-vectorspace of G-equivariant morphisms $X \to M^*$ into the dual module M^* of M. If X is a G-orbit, $X \simeq G/H$, we obtain

$$\mathrm{mult}_M(X) = \dim(M^*)^H \qquad \text{(Frobenius reciprocity)}.$$

Now let $C \subset \mathbf{g}$ be a conjugacy class and $\bar{C}$ its closure in $\mathbf{g}$. Since the complement $\bar{C} \setminus C$ is of codimention $\geqslant 2$ we have the following result due to Kostant ([Ko, 2.2 Proposition 9]; cf. [BK]):

PROPOSITION. *The closure $\bar{C}$ of the conjugacy class C is normal if and only if* $\mathrm{mult}_M(\bar{C}) = \mathrm{mult}_M(C)$ *for all irreducible representations M of G.*

5.2. For the proof of the normality of $\overline{C_{10}}$ we will need to know the multiplicities of the representations V and W of highest weight ω_1 and $2\omega_1$ in $k[C_{10}]$.

LEMMA. *For the representations V and W of highest weight ω_1 and $2\omega_1$ we have*

$$\operatorname{mult}_V(C_{10}) = 0 \qquad \textit{and} \qquad \operatorname{mult}_W(C_{10}) \leqslant 1.$$

Proof. We know that C_{10} is the conjugacy class of $x = x_2 + x'_2$ (notations of 2.1). By Frobenius reciprocity (5.1) we have to show that

$$V^H = 0 \qquad \text{and} \qquad \dim W^H \leqslant 1,$$

where $H = G_x$ is the centralizer of x. (Remember that V and W are selfdual.)

Let $U \subset G$ be the unipotent subgroup with Lie algebra

$$\operatorname{Lie} U = \mathbf{g}_{\alpha_1+\alpha_2} \oplus \mathbf{g}_{2\alpha_1+\alpha_2} \oplus \mathbf{g}_{3\alpha_1+2\alpha_2}.$$

We know that $U \subset H$ (Remark 2.3). Since U is normalized by the maximal torus T the fixed point set V^U is a sum of weight spaces. In fact, it is easy to see that

$$V^U = V_{\alpha_1+\alpha_2} \oplus V_{2\alpha_1+\alpha_2}.$$

Now the center Z of SL_3 belongs to H, too (Remark 2.3). Since every short root β is non-trivial on Z we obtain

$$V^H \subset (V^U)^Z = 0,$$

proving the first claim.

The second symmetric power $S^2(V)$ of V contains the irreducible representation W, which is of dimension 27, hence $S^2(V) \simeq k \oplus W$ as a G-module. It follows that the weights in W are $0, \alpha_1, \alpha_2, 2\alpha_1$ and their conjugates under the Weyl group, and their multiplicities are given by

$$\dim W_0 = 3, \qquad \dim W_{\alpha_1} = 2, \qquad \dim W_{\alpha_2} = 1, \qquad \dim W_{2\alpha_1} = 1.$$

It is easy to see that

$$W^U \subset W_0 \oplus W_{\alpha_1+\alpha_2} \oplus W_{2\alpha_1+\alpha_2} \oplus W_{2(\alpha_1+\alpha_2)} \oplus W_{2(2\alpha_1+\alpha_2)} \oplus W_{3\alpha_1+2\alpha_2}.$$

We claim that $W^U \cap W_0 = 0$, and therefore

$$W^U \subset W_{\alpha_1+\alpha_2} \oplus \cdots \oplus W_{3\alpha_1+2\alpha_2}.$$

In fact, if $w \in W_0$ is annihilated by $\mathbf{g}_\beta$, it is also annihilated by $\mathbf{g}_{-\beta}$. It follows for an element $w \in W^U \cap W_0$ that $\mathbf{g}_\beta \omega = 0$ for every short root β, hence w is annihilated by all of $\mathbf{g}$, and the claim follows.

Considering now the action of the center Z of SL_3 the same argument as above applies and shows that

$$W^H \subset (W^U)^Z \subset W_{3\alpha_1+2\alpha_2},$$

proving the second claim. ∎

6. Normality of $\overline{C_{10}}$

6.1. We want to calculate the schematic fibre of $0 \in \overline{C_{10}}$ under the finite surjective morphism $p_{10}: \overline{D_{10}} \to \overline{C_{10}}$ (4.2). By construction, $p_{10}^{-1}(0)$ is the schematic intersection $\overline{D_{10}} \cap V$, hence of the form $\operatorname{Spec} k[V]/J$, where J is a graded G-stable ideal whose radical is the homogeneous maximal ideal of $k[V]$.

6.2. Lemma. *The ideal J contains the invariant q and all homogeneous elements of degree $\geqslant 3$. In particular, the only representations possibly occurring in $k[V]/J$ are k, V, and W.*

Proof. Clearly $q \in J$ since q is the restriction of the non-degenerate quadratic invariant $\tilde{q} \in k[\mathbf{so}_7]^{SO_7}$ to $V \subset \mathbf{so}_7$, and $\tilde{q}$ vanishes on D_{10}. Furthermore, the elements of $\overline{D_{10}} \subset \mathbf{so}_7$ are endomorphisms of rank $\leqslant 2$ (see 4.1). Hence the 3×3-minors generate a $\mathbf{so}_7$-stable subvectorspace $M \subset k[\mathbf{so}_7]$ of homogeneous polynomials of degree 3 vanishing on $\overline{D_{10}}$. More precisely $\overline{D_{10}}$ is the zero set of M together with all homogeneous SO_7-invariant functions on $\mathbf{so}_7$. These invariants restrict to multiples of q on V (3.1). Since $\overline{D_{10}} \cap V = \{0\}$ this implies that q and the restriction $\bar{M} := M|_V$ generate an ideal of finite index in $k[V]$.

Now the zero fibre $q^{-1}(0) \subset V$ is the closure of the orbit of a highest weight vector, hence $k[q^{-1}(0)] \simeq \bigoplus_{i \geqslant 0} V_i$ as a G-module, where V_i is simple of highest weight $i\omega_1$ [VP, Theorem 2]. It follows that

$$k[V] \simeq k[q] \otimes \bigoplus_{i \geqslant 0} V_i$$

as a G-module because q is an irreducible polynomial and V a free $k[q]$-module (cf. [Ko]). In particular,

$$k[V]_0 = k, \qquad k[V]_1 \simeq V, \qquad k[V]_2 \simeq kq \oplus W, \qquad k[V]_3 \simeq qV \oplus V_3,$$

where $k[V]_d$ is the homogeneous component of $k[V]$ of degree d. Now $\bar{M} \subseteq k[V]_3$ is G-stable and $\bar{M} \not\subset qV$, since q and $\bar{M}$ generate an ideal of finite index, as we have seen above. Hence $\bar{M} \supseteq V_3$ and so

$$J \supseteq (q, \bar{M}) \supseteq \bigoplus_{i \geqslant 3} k[V]_i. \quad \blacksquare$$

6.3. *Proof of Normality.* Now we want to show that $\overline{C_{10}}$ is normal. Let $\eta : \widetilde{C_{10}} \to \overline{C_{10}}$ be the normalization. Since $\overline{D_{10}}$ is normal [KP3, Sect. 19 tables], we get a factorization

$$p_{10} : \overline{D_{10}} \xrightarrow{\widetilde{p_{10}}} \widetilde{C_{10}} \xrightarrow{\eta} \overline{C_{10}}$$

with a finite surjective morphism $\widetilde{p_{10}}$. In terms of coordinate rings this means that we have finite extensions

$$k[\overline{C_{10}}] \subset k[\widetilde{C_{10}}] \subset k[\overline{D_{10}}].$$

Next we show that $k[\widetilde{C_{10}}]$ is a direct summand of $k[\overline{D_{10}}]$. By Frobenius reciprocity (5.1), the second inclusion is of the form $k[G]^{H_2} \subset k[G]^{H_1}$ with subgroups $H_1 \subset H_2 \subset G$, and $k[G]^{H_1} \subset k[G]^{H_1^0} = k[G]^{H_2^0}$. Now there is a finite subgroup $F \subset H_2$ such that $H_2 = F \cdot H_2^0$, hence $k[G]^{H_2} = (k[G]^{H_2^0})^F$. This implies that $k[G]^{H_2}$ is a direct summand of $k[G]^{H_2^0}$, and therefore $k[\widetilde{C_{10}}]$ is a direct summand of $k[\overline{D_{10}}]$.

As a consequence of this we see that the schematic fibre $\eta^{-1}(0) \simeq \operatorname{Spec} R$ is given by a G-stable subalgebra $R \subset k[V]/J$ (notations of 6.1). We have to show that $R = k$. But $\operatorname{mult}_V(\widetilde{C_{10}}) = \operatorname{mult}_V(C_{10}) = 0$ by Lemma 5.2 and so V cannot occur in R. Also $\operatorname{mult}_W(\widetilde{C_{10}}) = \operatorname{mult}_W(C_{10}) \leqslant 1$ (Lemma 5.2) and $\operatorname{mult}_W(\overline{C_{10}}) \geqslant 1$ by the lemma below. This shows that R does not contain W either, proving our claim. $\blacksquare$

6.4. Lemma. *The map* $\mathbf{so}_7 \to M_7(k)$, $X \mapsto X^2$, *induces a non-trivial G-equivariant morphism* $\overline{C_{10}} \to V$. *In particular* $\operatorname{mult}_V(\overline{C_{10}}) \geqslant 1$.

Proof. It follows from Lemma 3.2 that the map $\mathbf{so}_7 \xrightarrow{?^2} M_7(k)$ is non-zero on C_{10}. Its image lies in Sym_7, the symmetric matrices in $M_7(k)$, and

$$\mathrm{Sym}_7 \simeq S^2V \simeq V \oplus k$$

as a G-module. If we compose the G-equivariant map

$$\lambda : \mathbf{g} \hookrightarrow \mathbf{so}_7 \xrightarrow{?^2} \mathrm{Sym}_7$$

with the projection onto k we obtain a quadratic invariant, hence a multiple of q_2, which vanishes on all nilpotents. It follows that the composition of λ with the other projection, the one onto V, induces a non-trivial covariant $\overline{C_{10}} \to V$. ∎

7. End of Proof of Theorem 1.2

We first remark that all non-nilpotent conjugacy classes have a normal closure with rational singularities: Every such closure is a G-fibre bundle over a semisimple class, whose fibre is the closure of a nilpotent class in some strict Levi subalgebra of $\mathbf{g}$ (see [Sl, Lemma 3.10]). The regular class C_{12} has a normal closure with rational singularities [Ko, He1]) and the singularity of $\overline{C_{12}}$ in C_{10} is of type D_4 (see [Sl]). The minimal class C_6 has a normal closure $\overline{C_6} = C_6 \cup \{0\}$ with rational singularities [Ke].

In the preceding section we have already shown that $\overline{C_{10}}$ is normal. In addition, we have seen in Section 4 that $C_{10} \subset D_{14}$ and that $C_8 \subset D_{12}$ (Remark 4.3). Since $\overline{C_{10}}$ is normal and since the codimensions of C_8 in $\overline{C_{10}}$ and of D_{12} in $\overline{D_{14}}$ are equal we can apply [KP3, Corollary 13.3]. This result states that $\overline{C_{10}}$ has in C_8 a singularity of the same type as $\overline{D_{14}}$ in D_{12}, which is a singularity of type A_1 [KP3, Sect. 19 tables].

Now $\overline{D_{10}}$ is normal with rational singularities (4.1). Applying Boutot's theorem [Bo] to the finite morphism $p_{10}: \overline{D_{10}} \to \overline{C_{10}}$ it follows that $\overline{C_{10}}$ has rational singularities, too. (In fact we have seen in 6.3 that the coordinate ring $k[\overline{C_{10}}]$ is a direct summand of $k[\overline{D_{10}}]$.)

Finally, $\overline{D_8} = D_8 \cup \{0\}$ is normal with an isolated rational singularity in 0 (4.1). Therefore the bijective morphism $p_8: \overline{D_8} \to \overline{C_8}$ (Proposition 4.2) is the normalization. This finishes the proof of the theorem.

8. Survey of Results on the Normality Problem

In this last section we give a brief summary of what is known about normality of closures of conjugacy classes in reductive groups.

8.1. *Reduction to the Simple Case.* Let G be a reductive group and $\mathbf{g} = \operatorname{Lie} G$ its Lie algebra. A conjugacy class C in G (or in $\mathbf{g}$) is of the form $G *^H C'$, where $H \subseteq G$ is a Levi subgroup and $C' \subseteq \operatorname{Lie} H$ a nilpotent conjugacy class, and the closure $\bar{C}$ of C is isomorphic to $G *^H \bar{C}'$ (cf. [Sl, II.3.10] or [KP3, 0.2]). Furthermore, a class $C \subset \mathbf{g}_1 \oplus \mathbf{g}_2$ is a product $C_1 \times C_2$ of two classes, and its closure is $\overline{C_1} \times \overline{C_2}$. This reduces the problem to the study of *nilpotent* classes in *simple* Lie algebras.

Recall that a semisimple group contains only finitely many nilpotent conjugacy classes, and every conjugacy class has even dimension (cf. [BC]).

8.2. *Some General Normality Results.* Kostant showed in his fundamental paper [Ko] that the *nilpotent cone* in $\mathbf{g}$, which is the closure of the *regular nilpotent class* C_{reg}, is a normal complete intersection, and Hesselink proved that it has rational singularities [He1].

For the *minimal class* $C_{\min}$, i.e., the orbit of highest weight vectors in $\mathbf{g}$, a general result of Vinberg and Popov [VP] implies that the closures $\bar{C}_{\min} = C_{\min} \cup \{0\}$ are normal, and it follows from Kempf [Ke] that it has rational singularities.

Using Serre's normality criterion Hesselink showed that certain classes in SL_n can be obtained as normal complete intersections in determinantal subvarieties of $\mathbf{g}$ defined by rank conditions [He2, 1.2]. Hence they are *normal and Cohen–Macaulay*. In addition, using resolution of singularities and [Ke] he discovered several "small" nilpotent classes in $\mathbf{sl}_n$, $\mathbf{so}_n$ and $\mathbf{sp}_n$, and in the exceptional Lie algebras besides the regular and the minimal class which have a normal closure with rational singularities, e.g., one in F_4, one in E_6, two in E_7, and one in E_8.

Brieskorn studied the singularity of the nilpotent cone in the *subregular class* (this is the only class of codimension 2) in case of a simple Lie algebra of type A, D, and E and showed that it is equivalent to a simple surface singularity of corresponding type (cf. Slodowy's book [Sl], where this is extended to all simple Lie algebras).

8.3. *Special Linear Groups.* In these groups the closure of every conjugacy class is normal and has rational singularities [KP1]. There is a simple algorithm in terms of the partition associated to a nilpotent class C to determine the classes occurring in the closure $\bar{C}$. Also one can read from the partition the type of the singularity of $\bar{C}$ in the open classes in the boundary $\partial C := \bar{C} \setminus C$, the so-called *minimal singularities* of C [KP2]. This generalizes the results of Brieskorn and Slodowy (8.2).

8.4. *Orthogonal and Symplectic Groups.* For these groups and their Lie algebras there exist conjugacy classes with non-normal closures [KP3]. Again the partition of a nilpotent conjugacy class C determines which classes appear in the closure $\bar{C}$, the type of the minimal singularities, and whether $\bar{C}$ is normal or not, except for the so-called *very even* classes. (These are the conjugacy classes in $\mathbf{so}_{4n}$ which are not stable under the full orthogonal group O_{4n}.) There are partial results about Cohen–Macaulayness and rational singularities.

It is interesting to remark at this point that the non-normal closures $\bar{C}$ are always *branched in codimension* 2, i.e., there is a class $D \subset \bar{C}$ of codimension 2 which undergoes a 2-fold covering in the normalization $\eta: \tilde{C} \to \bar{C}$. As we have seen in 1.2 this is not the case for the class C_8 in G_2.

8.5. *Branched Non-normality (Beynon and Spaltenstein).* In the paper [BS, Sect. 5(E)] one finds the following result (based on a remark of Verdier and Brylinski):

PROPOSITION. *Let $x, y \in \mathfrak{g}$ be nilpotent elements and let $\mathscr{B}_y$ be the fibre of y under the Springer resolution of singularities of the nilpotent cone of $\mathfrak{g}$. Denote by ρ_x the Weyl-group representation corresponding to x (and the trivial character of the component group G_x/G_x^0) under the Springer-correspondence. Then*

$$\operatorname{mult}_{\rho_x} H^{2\beta(x)}(\mathscr{B}_y) = \# \eta^{-1}(y),$$

where $\beta(x) = \dim \mathscr{B}_x$ and $\eta: \tilde{C} \to \bar{C}$ is the normalisation.

It follows that these multiplicities determine the inclusion behavior of closures of nilpotent conjugacy classes in $\mathfrak{g}$ and allow one to detect all classes with *branched non-normal closures.* The multiplicities have been calculated by Shoji [Sh] for F_4 and by Beynon and Spaltenstein [BS] for E_6, E_7, and E_8. Inspecting their tables one finds the following classes with a branched non-normal closure (we use the notations of Bala and Carter [BC]):

$F_4: A_1 + B_2, C_3$

$E_6: A_4, 2A_2, A_2 + A_1$

$E_7: D_6(a_1), A_5'', A_4, A_3 + A_2, D_4(a_1) + A_1$

$E_8: E_7(a_1), E_6, E_6(a_1), E_7(a_4), A_6, D_6(a_1), D_5 + A_1, E_7(a_5), A_4,$
$A_3 + A_2, D_4, D_4(a_1), A_3 + A_1, 2A_2 + A_1$

8.6. *The Method of Richardson.* The paper [Ri] contains a method to calculate the rank of the quotient map $\pi: \mathfrak{g} \to k^r$ in a given nilpotent element $x \in \mathfrak{g}$. It is easy to see that $\operatorname{rank} d\pi_x = \operatorname{mult}_{\mathfrak{g}}(\overline{C_x})$ (notations 5.1), where C_x is the conjugacy class of x [Ri, Proposition 2.7]. On the other hand, it is shown in [BK] that under certain assumptions about the stabilizer G_x, the multiplicities are constant along the sheets containing the class C_x in case $\overline{C_x}$ is normal. Thus comparing rank $d\pi_x$ with the rank of the quotient map in the corresponding element of the Levi subalgebra associated to the sheet, Richardson obtains a sufficient criterion for non-normality. He detects in this way

4 classes in E_6, 5 classes in E_7 and 11 classes in E_8,

which have a non-normal closure with a *bijective normalisation.*

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